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LETTER TO THE EDITOR

A determinantal formula for the GOE Tracy–Widom distribution

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Abstract
Investigating the long-time asymptotics of the totally asymmetric simple exclusion process, Sasamoto obtains rather indirectly a formula for the Tracy–Widom distribution for the Gaussian orthogonal ensemble. We establish that his novel formula indeed agrees with more standard expressions.

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1. Introduction

The Gaussian orthogonal ensemble (GOE) of random matrices is a probability distribution on the set of $N \times N$ real symmetric matrices defined through

$$Z^{-1}\,\mathrm{e}^{-\mathrm{Tr}(H^2)/2N}\,\mathrm{d}H. \tag{1}$$

Z is the normalization constant and $\mathrm{d}H = \prod_{1\leqslant i\leqslant j\leqslant N}\mathrm{d}H_{i,j}$. The induced statistics of eigenvalues can be studied through the method of Pfaffians. Of particular interest for us is the statistics of the largest eigenvalue, E_1. As proved by Tracy and Widom [8], the limit

$$\lim_{N\to\infty}\mathbb{P}(E_1 \leqslant 2N + sN^{1/3}) = F_1(s) \tag{2}$$

exists, $\mathbb{P}$ being our generic symbol for probability of the event in parenthesis. F_1 is called the GOE Tracy–Widom distribution function. Following [3], it can be expressed in terms of a Fredholm determinant in the Hilbert space $L^2(\mathbb{R})$ as follows:

$$F_1(s)^2 = \det(\mathbb{1} - P_s(K + |g\rangle\langle f|)P_s), \tag{3}$$

where K is the Airy kernel defined through

$$\begin{aligned} K(x,y) &= \int_{\mathbb{R}_+}\mathrm{d}\lambda\,\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda), \qquad g(x) = \mathrm{Ai}(x), \\ f(y) &= 1 - \int_{\mathbb{R}_+}\mathrm{d}\lambda\,\mathrm{Ai}(y+\lambda), \end{aligned} \tag{4}$$

and P_s is the projection onto the interval $[s,\infty)$.

The GOE Tracy–Widom distribution $F_1(s)$ turns up also in the theory of one-dimensional growth process in the KPZ universality class, KPZ standing for Kardar–Parisi–Zhang [4]. Let us denote the height profile of the growth process at time t by $h(x,t)$, either $x \in \mathbb{R}$ or $x \in \mathbb{Z}$. One then starts the growth process with flat initial conditions, meaning $h(x,0) = 0$, and considers the height above the origin $x = 0$ at growth time t. For large t, it is expected that

$$h(0,t) = c_1 t + c_2 t^{1/3} \xi_1. \tag{5}$$

Here c_1 and c_2 are constants depending on the details of the model and ξ_1 is a random amplitude with

$$\mathbb{P}(\xi_1 \leqslant s) = F_1(s). \tag{6}$$

For the polynuclear growth (PNG) model, the height $h(0,t)$ is related to the length of the longest increasing subsequence of symmetrized random permutations [5], for which Baik and Rains [1] indeed prove the asymptotics (5) and (6); see [2] for further developments along this line. Very recently Sasamoto [6] succeeded in proving the corresponding result for the totally asymmetric simple exclusion process (TASEP). If $\eta_j(t)$ denotes the occupation variable at $j \in \mathbb{Z}$ at time t, then the TASEP height is given by

$$h(j,t) = \begin{cases} 2N_t + \sum_{i=1}^{j} (1 - 2\eta_i(t)) & \text{for} \quad j \geqslant 1, \\ 2N_t & \text{for} \quad j = 0, \\ 2N_t - \sum_{i=j+1}^{0} (1 - 2\eta_i(t)) & \text{for} \quad j \leqslant -1, \end{cases} \tag{7}$$

with N_t denoting the number of particles which passed through the bond (0, 1) up to time t. The flat initial condition for the TASEP is . . . 0 1 0 1 0 1 . . . . For technical reasons Sasamoto takes instead . . . 0 1 0 1 0 0 0 0 0 . . . and studies the asymptotics of $h(-3t/2, t)$ for large t with the result

$$h(-3t/2, t) = \tfrac{1}{2} t + \tfrac{1}{2} t^{1/3} \xi_{\mathrm{SA}}. \tag{8}$$

The distribution function of the random amplitude ξ_{SA} is

$$\mathbb{P}(\xi_{\mathrm{SA}} \leqslant s) = F_{\mathrm{SA}}(s) \tag{9}$$

with

$$F_{\mathrm{SA}}(s) = \det(\mathbb{1} - P_s A P_s). \tag{10}$$

Here A has the kernel $A(x,y) = \frac{1}{2}\mathrm{Ai}((x+y)/2)$ and, as before, the Fredholm determinant is in $L^2(\mathbb{R})$.

The universality hypothesis for one-dimensional growth processes claims that in the scaling limit, up to model-dependent coefficients, the asymptotic distributions are identical. In particular, since (5) is proved for PNG, the TASEP with flat initial conditions should have the same limit distribution function, to say

$$F_{\mathrm{SA}}(s) = F_1(s). \tag{11}$$

Our contribution provides a proof for (11).

2. The identity

As written above, the s-dependence sits in the projection P_s. It will turn out to be more convenient to transfer the s-dependence into the integral kernel. From now on, the determinants are understood as Fredholm determinants in $L^2(\mathbb{R}_+)$ with scalar product $\langle \cdot, \cdot \rangle$. Thus, whenever we write an integral kernel like $A(x,y)$, the arguments are understood as $x \geqslant 0$ and $y \geqslant 0$.

Let us define the operator $B(s)$ with kernel

$$B(s)(x, y) = \mathrm{Ai}(x + y + s). \tag{12}$$

By [7], $\|B(s)^2\| < 1$ and clearly $B(s)$ is symmetric. Thus also $\|B(s)\| < 1$ for all s. $B(s)$ is trace class with both positive and negative eigenvalues. Shifting the arguments in (10) by s, one notes that

$$F_{\mathrm{SA}}(s) = \det(\mathbb{1} - B(s)). \tag{13}$$

Applying the same operation to (3) yields

$$F_1(s)^2 = \det(\mathbb{1} - B(s)^2 - |g\rangle\langle f|), \tag{14}$$

with

$$\begin{aligned} g(x) &= \mathrm{Ai}(x + s) = (B(s)\delta)(x), \\ f(y) &= 1 - \int_{\mathbb{R}_+} \mathrm{d}\lambda\, \mathrm{Ai}(y + \lambda + s) = ((\mathbb{1} - B(s))1)(y). \end{aligned} \tag{15}$$

Here δ is the δ-function at $x = 0$ and 1 denotes the function $1(x) = 1$ for all $x \geqslant 0$. δ and 1 are not in $L^2(\mathbb{R}_+)$. Since the kernel of $B(s)$ is continuous and has super-exponential decay, the action of $B(s)$ is unambiguous.

Proposition 1. *With the above definitions, we have*

$$\det(\mathbb{1} - B(s)) = F_1(s). \tag{16}$$

Proof. For simplicity, we suppress the explicit s-dependence of B. We rewrite

$$\begin{aligned} F_1(s)^2 &= \det((\mathbb{1} - B)(\mathbb{1} + B - |B\delta\rangle\langle 1|)) \\ &= \det(\mathbb{1} - B)\det(\mathbb{1} + B)(1 - \langle \delta, B(\mathbb{1} + B)^{-1}1\rangle) \\ &= \det(\mathbb{1} - B)\det(\mathbb{1} + B)\langle \delta, (\mathbb{1} + B)^{-1}1\rangle, \end{aligned} \tag{17}$$

since $1 = \langle \delta, 1\rangle$. Thus, we have to prove that

$$\det(\mathbb{1} - B) = \det(\mathbb{1} + B)\langle \delta, (\mathbb{1} + B)^{-1}1\rangle. \tag{18}$$

Taking the logarithm on both sides,

$$\ln\det(\mathbb{1} - B) = \ln\det(\mathbb{1} + B) + \ln\langle \delta, (\mathbb{1} + B)^{-1}1\rangle, \tag{19}$$

and differentiating it with respect to s results in

$$-\mathrm{Tr}\left((\mathbb{1} - B)^{-1}\frac{\partial}{\partial s}B\right) = \mathrm{Tr}\left((\mathbb{1} + B)^{-1}\frac{\partial}{\partial s}B\right) + \frac{\frac{\partial}{\partial s}\langle \delta, (\mathbb{1} + B)^{-1}1\rangle}{\langle \delta, (\mathbb{1} + B)^{-1}1\rangle}, \tag{20}$$

where we used

$$\frac{\mathrm{d}}{\mathrm{d}s}\ln(\det(T)) = \mathrm{Tr}\left(T^{-1}\frac{\partial}{\partial s}T\right). \tag{21}$$

Since $B(s) \to 0$ as $s \to \infty$, the integration constant for (20) vanishes and we have to establish that

$$-2\,\mathrm{Tr}\left((\mathbb{1} - B^2)^{-1}\frac{\partial}{\partial s}B\right) = \frac{\frac{\partial}{\partial s}\langle \delta, (\mathbb{1} + B)^{-1}1\rangle}{\langle \delta, (\mathbb{1} + B)^{-1}1\rangle}. \tag{22}$$

Define the operator $D = \frac{\mathrm{d}}{\mathrm{d}x}$. Then, using the cyclicity of the trace and lemma 2,

$$\begin{aligned} -2\,\mathrm{Tr}\left((\mathbb{1} - B^2)^{-1}\frac{\partial}{\partial s}B\right) &= -2\,\mathrm{Tr}((\mathbb{1} - B^2)^{-1}DB)) \\ &= \langle \delta, (\mathbb{1} - B^2)^{-1}B\delta\rangle. \end{aligned} \tag{23}$$

Using lemma 3 and $D1 = 0$, one obtains

$$\langle \delta, \frac{\partial}{\partial s}(\mathbb{1} + B)^{-1} 1\rangle = \langle \delta, (\mathbb{1} - B^2)^{-1} B\delta\rangle\langle \delta, (\mathbb{1} + B)^{-1} 1\rangle. \tag{24}$$

Thus, (22) follows from (23) and (24). □

Lemma 2. *Let A be a symmetric, trace class operator with smooth kernel and let $D = \frac{\mathrm{d}}{\mathrm{d}x}$. Then,*

$$2\,\mathrm{Tr}(DA) = -\langle \delta, A\delta\rangle, \tag{25}$$

where DA is the operator with kernel $\frac{\partial}{\partial x}A(x, y)$.

Proof. The claim follows from spectral representation of A and the identity

$$\int_{\mathbb{R}_+} \mathrm{d}x f'(x) f(x) = -f(0) f(0) - \int_{\mathbb{R}_+} \mathrm{d}x\, f(x) f'(x). \tag{26}$$

□

Lemma 3. *It holds that*

$$\frac{\partial}{\partial s}(\mathbb{1} + B)^{-1} = (\mathbb{1} - B^2)^{-1} BD + (\mathbb{1} - B^2)^{-1}|B\delta\rangle\langle\delta(\mathbb{1} + B)^{-1}|. \tag{27}$$

Proof. First notice that $\frac{\partial}{\partial s}B \equiv \dot{B} = DB$. For any test function f,

$$\begin{aligned}(\dot{B} f)(x) &= \int_{\mathbb{R}_+} \mathrm{d}y \partial_y \,\mathrm{Ai}(x + y + s) f(y) \\ &= -\mathrm{Ai}(x + s) f(0) - \int_{\mathbb{R}_+} \mathrm{d}y\, \mathrm{Ai}(x + y + s) f'(y).\end{aligned} \tag{28}$$

Thus, using the notation $P = |B\delta\rangle\langle\delta|$, one has

$$DB = -BD - P. \tag{29}$$

Since $\|B\| < 1$, we can expand $\frac{\partial}{\partial s}(\mathbb{1} + B)^{-1}$ in a power series and get

$$\frac{\partial}{\partial s}(\mathbb{1} + B)^{-1} = \sum_{n\geqslant 1}(-1)^n \frac{\partial}{\partial s}B^n = \sum_{n\geqslant 1}(-1)^n \sum_{k=0}^{n-1} B^k D B^{n-k}. \tag{30}$$

Using recursively (29), we obtain

$$\begin{aligned}\sum_{k=0}^{n-1} B^k D B^{n-k} &= -\frac{1 - (-1)^n}{2} B^n D + \sum_{j=0}^{n-1}\sum_{k=j}^{n-1} (-1)^{j+1} B^k P B^{n-k-1} \\ &= -\frac{1 - (-1)^n}{2} B^n D + \sum_{k=0}^{n-1} \frac{1 + (-1)^k}{2} B^k P B^{n-k-1}.\end{aligned} \tag{31}$$

Inserting (31) into (30) and exchanging the sums results in

$$\begin{aligned}\frac{\partial}{\partial s}(\mathbb{1} + B)^{-1} &= \sum_{n\geqslant 1} B^{2n+1} D + \sum_{k\geqslant 0}\sum_{n\geqslant k+1} \frac{1 + (-1)^k}{2} B^k P(-B)^{n-(k+1)} \\ &= (\mathbb{1} - B^2)^{-1} BD + (\mathbb{1} - B^2)^{-1} P(\mathbb{1} + B)^{-1}.\end{aligned} \tag{32}$$

□

3. Outlook

The asymptotic distribution of the largest eigenvalue is also known for Gaussian unitary ensemble of Hermitian matrices ($\beta = 2$) and Gaussian symplectic ensemble of quaternionic symmetric matrices ($\beta = 4$). As just established, for $\beta = 1$,

$$F_1(s) = \det(\mathbb{1} - B(s)), \tag{33}$$

and, for $\beta = 2$,

$$F_2(s) = \det(\mathbb{1} - B(s)^2), \tag{34}$$

which might indicate that $F_4(s)$ equals $\det(\mathbb{1} - B(s)^4)$. This is however incorrect, since the decay of $\det(\mathbb{1} - B(s)^4)$ for large s is too rapid. Rather, one has

$$F_4(s/\sqrt{2}) = \tfrac{1}{2}(\det(\mathbb{1} - B(s)) + \det(\mathbb{1} + B(s))). \tag{35}$$

This last identity is obtained as follows. Let $U(s) = \frac{1}{2}\int_s^\infty q(x)\,\mathrm{d}s$ with q the unique solution of the Painlevé II equation $q'' = sq + 2q^3$ with $q(s) \sim \mathrm{Ai}(s)$ as $s \to \infty$. Then, the Tracy–Widom distributions for $\beta = 1$ and $\beta = 4$ are given by

$$F_1(s) = \exp(-U(s))F_2(s)^{1/2}, \qquad F_4(s/\sqrt{2}) = \cosh(U(s))F_2(s)^{1/2}, \tag{36}$$

see [8]. Thus, $F_4(s/\sqrt{2}) = \frac{1}{2}(F_1(s) + F_2(s)/F_1(s))$, from which (35) is deduced.

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